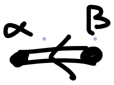
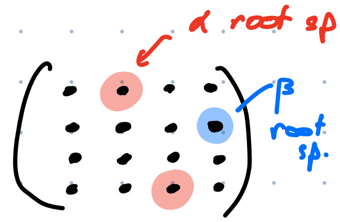
Lecture 19.

$\begin{matrix} A & B \\ C = -A^T & \end{matrix}$ $\quad B = B^T \quad C = C^T$

$Sp(4,\mathbb{C}) = G$ type B_2 

Borel $= \begin{pmatrix} \begin{matrix} * & * \\ 0 & * \end{matrix} & \begin{matrix} * & * \\ * & * \end{matrix} \\ & \begin{matrix} -* & 0 \\ -* & -* \end{matrix} \end{pmatrix} = \mathrm{Stab}([e_1], [e_1, e_2])$

Coords $p_1\, p_2\, q_1\, q_2$ $\quad \omega(e_i, e_j) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
$e_1\, e_2\, e_3\, e_4$



Full flag variety: $G/B = \{ (F_1, F_2, F_1^{\perp}) \mid \omega|_{F_2} = 0 \}$

Minimal flag var: $(\Theta = \Delta \setminus \{\gamma\})$

$G/P_{\Delta - \{\alpha\}} = \mathbb{CP}^3$

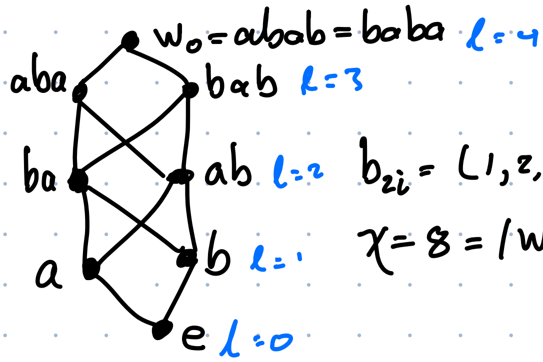
$G/P_{\Delta - \{\beta\}} = \mathcal{L}ag(\mathbb{C}^4, \omega)$ isotropic (Lagrangian) lines in $\mathbb{C}^4$.

$W \cong \underbrace{S_2}_{\text{permutes bits}} \ltimes \underbrace{(\mathbb{Z}/2)^2}_{\text{binary vectors}}$

s_α corresp to the generator of S_2, i.e. $(\cdot, \cdot)$

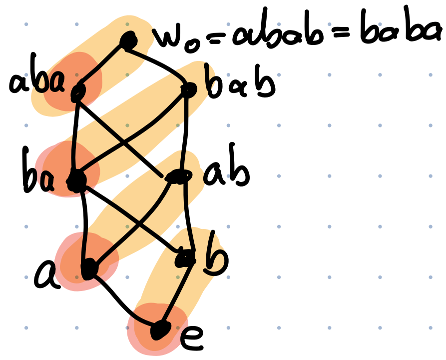
s_β corresp to flipping one bit i.e. $(\cdot,\cdot) \to (\bar{\cdot}, \cdot)$

Bruhat order



$b_{2i} = (1,2,2,2,1)$

$\chi = 8 = |W|$



$\mathbb{CP}^3$: $\Theta = \Delta - \{\alpha\} = \{\beta\}$

Cosets of $W_\Theta = \langle s_\beta \rangle = \langle b \rangle$

Bruhat diag of $W/W_{P_{\Delta-\{\alpha\}}}$:

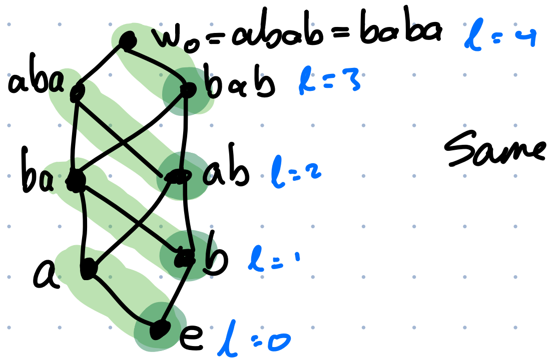
$aba\,W_{P_{\Delta-\{\alpha\}}}$ $\quad l=3$

$ba\,W_{P_{\Delta-\{\alpha\}}}$ $\quad l=2$

$a\,W_{P_{\Delta-\{\alpha\}}}$ $\quad l=1$

$e\,W_{P_{\Delta-\{\alpha\}}}$ $\quad l=0$

$\mathrm{Lag}(\mathbb{C}^3)$: $\Theta = \Delta - \{\beta\} = \{\alpha\}$ $\quad W_\Theta = \langle s_\alpha \rangle = \langle a \rangle$



Same betti numbers as $\mathbb{C}P^3$!

Aside. Why W/W_P give cells in G/P.

$\pi: G/B \to G/P$ has fibers P/B.

We've seen that a cell in G/B is the image of a product of unipotent groups.

$$U_{\alpha_1} \times \cdots \times U_{\alpha_\ell} \to G/B$$
$$(u_1, \ldots, u_\ell) \mapsto u_1 u_2 \cdots u_\ell w B$$

Here α_i are the roots flipped by w.

In G/P, however, some of the U_{α_i} stabilize wP

$\Leftrightarrow$ $w^{-1} U_{\alpha_i} w$ stabilizes P

$\Leftrightarrow$ $w^{-1} U_{\alpha_i} w$ is in P

$\Leftrightarrow$ α_i is in the $\mathbb{Z}$-span of Θ

The min len rep of a class wW_P doesn't flip anything in the $\mathbb{Z}$-span of Θ, so its associated map

$$\prod U_{\alpha_i} \to G/P \quad \text{is injective}$$

Grassmannians

$G/P = Gr(d,n)$. $G = SL_n\mathbb{C}$ type A_{n-1} $\quad W = \langle \alpha_1 \ldots \alpha_{n-1} \rangle = \mathrm{Sym}_n$

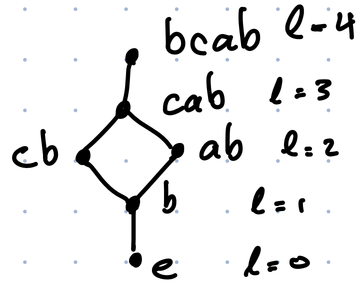
P = stab of flag $[e_1 \ldots e_d] \subset \mathbb{C}^n$. $\quad \begin{pmatrix} \boxed{d} & \\ & \boxed{n-d} \end{pmatrix}$

$W_P = \langle \alpha_1 \cdots \alpha_{d-1}, \alpha_{d+1}, \ldots, \alpha_{n-1} \rangle \simeq \mathrm{Sym}_d \times \mathrm{Sym}_{n-d}$

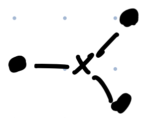
$[\sigma] \in W/W_P$ min len rep: $\sigma = (\sigma(1), \dots, \sigma(n))$ where $1, 2, \dots, d$ and $d+1, \dots, n$ appear in ascending order left to right.
$\leftrightarrow$ decompose $1, \dots, n$ into two subsets.

e.g. $Gr(2,4)$. $W = \langle a,b,c \rangle$, $W_P = \langle a,c \rangle$

1	2	3	4	e
1	3	2	4	b
1	3	4	2	cb
3	1	2	4	ab
3	1	4	2	cab
3	4	1	2	bcab



Middle dim Schubert cells:
Planes containing $[e_1]$
& Planes contained $[e_1 e_2 e_3]$

<u>Hard exercise</u>. Compute the betti numbers of G/P for G of type D_4 and $P =$  (So $G/B \to G/P$ has fibers $(\mathbb{CP}^1)^3$)
$\llcorner SO(8, \mathbb{C})$

<u>Cohomology</u> $H^*(G/B)$

$$H^*(G/B) \cong H^*(K/T)$$

$$\begin{tikzcd} K \arrow[r] \arrow[d] & EK \arrow[d] \\ \bullet \arrow[r] & BK \end{tikzcd}$$

$\downarrow$ quot by T

$$\begin{tikzcd} K/T \arrow[r] \arrow[d] & BT \arrow[d] \\ \bullet \arrow[r] & BK \end{tikzcd}$$

This diagram is a <u>pushout</u>.

$$H^*(K/T) \cong H^*(BT) \otimes_{H^*(BK)} H^*(\bullet)$$